

11.1-11.4

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Defn: • Let $I \subseteq [0, 1]$.

$$I_+ = \{0\} \cup \{j \in [0, 1] \mid j = \text{Sum of elts. of } I\}$$

$$\bullet \quad LCT_n(I) := LCT_n(I, \mathbb{N}) = \{\text{lct}(X, \Delta; M) \mid (X, \Delta) \text{ lc dim } n$$

$$\text{coeff}(\Delta) \subseteq I$$

$$M = \text{Eff. } \mathbb{Z}\text{-divisors}\}$$

Thm 1.11: "Accumulation points of LCT in $\dim n = LCT$ in $\dim n-1$ "

$$I \subseteq [0, 1] \text{ s.t. } \bullet \quad I = I_+$$

• 1 is the only accum. pt. of I

Then:

$$\text{Accum. pts of } LCT_n(I) = LCT_{n-1}(I) - \{1\}$$

In particular, $I \subseteq \mathbb{Q} \Rightarrow \text{Accum. pts} \subseteq \mathbb{Q}$

Pf sketch:

- Given (X_i, Δ_i) , want to understand limit of lct.
- Show $\{\text{lct's}\} = \{\text{Numerically trivial thresholds}\}$
"N_n(I)"
- Can assume (X_i, Δ_i) is log CY, lc, projective
- Mostly use induction either by applying adjunction
or by running MMP and then taking gen. fiber



Defn 11.1: • $D(I) := \{a \leq 1 \mid a = \frac{m-1+f}{m}, m \in \mathbb{N}, f \in I_+\}$

• Fix $c \in [0, 1]$

$$D_c(I) := \left\{ a \leq 1 \mid a = \frac{m-1+f+kc}{m}, m \in \mathbb{N}, f \in I_+, k \in \mathbb{N} \right\} \subseteq D(I \cup \{c\})$$

• $N_n(I, c) = \{(X, \Delta) \text{ log CY, lc, projective of dim } n \text{ s.t.}$

$$\begin{array}{c} \Delta = B + C \\ \text{In } D(I) \quad \text{In } D_c(I), \neq 0 \end{array}$$

- $N_n(I) = \{c \in [0, 1] \mid \eta_n(I, c) \text{ non-empty}\}$

Lemma 11.2: (1) $LCT_n(I) \subseteq LCT_{n+1}(I)$

(2) $N_n(I) \subseteq N_{n+1}(I)$

(3) $f \in I_+$, $k \in \mathbb{N}$. Then:

$$c = \frac{1-f}{k} \in N_n(I)$$

Pf: (1) Given (X, Δ) of dim n , $(X \times C, \Delta \times C)$ for smooth curve C

$$\text{lct}(X, \Delta; M) = \text{lct}(X \times C, \Delta \times C; M \times C)$$

$$\Rightarrow LCT_n(I) \subseteq LCT_{n+1}(I)$$

(2) Same proof works.

Take $C = \text{Elliptic curve}$ in above proof to ensure $X \times C, \Delta \times C$ log CY.

(3) Suffices to do for $n=1$. Explicit example:

[Take $X = \mathbb{P}^1$, $\Delta = B + C$ where $B = f P + f Q$, $C = 2kcR$ $P, Q, R \in \mathbb{P}^1]$



Defn 11.3: Want to restrict further the class of varieties we consider.

- $\eta_n(I, c) \supseteq K_n(I, c) :=$ Additionally $\begin{cases} (X, \Delta) \text{ klt} \\ X \text{ Q-factorial} \\ \varsigma(X) = 1 \end{cases}$

- $K_n(I) = \{c \in [0, 1] \mid K_m(I, c) \text{ non-empty for some } m \leq n\}$

Lemma 11.4: $N_n(I \cup \{1\}) = K_n(I)$

[In particular, $N_n(I) = N_n(I \cup \{1\}) = K_n(I)$]

Pf: • Clearly $K_n(I) \subseteq N_n(I \cup \{1\})$ true. [$K_n \subseteq \eta_n$ $\forall n$]

• NTS: $N_n(I \cup \{1\}) \subseteq K_n(I)$

Let $c \in N_n(I \cup \{1\})$. i.e. $\exists (X, \Delta) \in \eta_n(I \cup \{1\}, c)$

$$\Delta = A + B + C$$

Coeff = 1 $\in D(I)$ $D_c(I)$

Need to produce $(Y, \Gamma) \in k_m(I, c)$ i.e. (Y, Γ) of dim $m \leq n$,
 klt,
 Q-factorial
 $\varsigma(Y) = 1$

- Take a dlt modification of (X, Δ) to assume X Q-factorial.

- If A, C intersect, apply adjunction to a comp. S of A .

$$(S, \Theta) \in \eta_{n-1}(I \cup \{1\}, c) \Rightarrow S \in N_{n-1}(I \cup \{1\}) = K_{n-1}(I) \subseteq K_n(I)$$

Induction

$$D_c(D(I)) = D_c(I)$$

- Run $K_X + A + B - \text{MMP}$. End with a MFS $f: X \rightarrow Z$

- If $\dim Z > 0$, general fiber $(x_z, \Delta_z) \in \eta_K(I \cup \{1\}, c)$. Done by induction
- If $\dim Z = 0$, $\varsigma(X) = 1$. $A = 0 \Rightarrow (X, \Delta)$ klt. ✓
 A, C intersect. ✓



11.5-11.6

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Prop 11.5: ($\text{lct in dim } n+1 = \text{Num. trivial threshold in dim } n$)

$$I = I_+$$

$$\text{LCT}_{n+1}(I) = N_n(I)$$

Pf: (1) $\text{LCT}_{n+1}(I) \subseteq N_n(I)$:

- Let $c \in \text{LCT}_{n+1}(I)$. Then $(X, \Delta + cM)$. $c = \text{lct}(X, \Delta; M)$
- Have done something similar in Global to Local ACC.
- Can assume V is the only non-klt center of $(X, \Delta + cM)$
- If $V = \text{comp. of } M$, coeff of $V = 1$ in $\Delta + cM$

$$fV + cKV = V$$

$$\Rightarrow c = \frac{1-f}{K} \in N_n(I) \text{ by Lemma 11.2, (3).}$$

- O/w, take a dlt mod. $f: Y \rightarrow X$ of $(X, \Delta + cM)$:

$$K_Y + T + \underline{\Delta'} + \underline{cM'} = f^*(K_X + \Delta + cM)$$

- Choose comp. S of T which maps to V [3.3.1 (4)]

- Adjunction \Rightarrow Get (S, Θ) where

$$(K_Y + T + \underline{\Delta'} + \underline{cM'})|_S = K_S + \underline{\Theta}$$

$$\Rightarrow \Theta = A + B + C$$

Coeff 1 $\in D(I)$ $\in D_c(I)$

Almost implies $(S, \Theta) \in \eta_{n-1}(I \cup \{V\}, c)$

- A general fiber of $S \rightarrow V$, (S_v, Θ_v) is log CY.

$$(S_v, \Theta_v) \in \eta_k(I \cup \{V\}, c)$$

$$\Rightarrow c \in N_k(I) \subseteq N_n(I).$$



(2) $N_n(I) \subseteq \text{LCT}_{n+1}(I)$:

- Given n -dim variety, going to take cone over it.
- Let $c \in N_n(I)$. Have $(X, \Delta = \sum d_i \Delta_i) \in K_m(I, c)$ for $m \leq n$. Thus (X, Δ) klt, \mathbb{Q} -factorial, $\beta(X) = 1$
- If $m < n$, done by induction. Thus $\dim X = n$.

Thus (X, Δ) klt, \mathbb{Q} -factorial, $s(X) = 1$

- If $m < n$, done by induction. Thus $\dim X = n$.
- Construct cone (Y, Γ) over (X, Δ) w.r.t. ample $-K_X$
- $d_i = \frac{m_i - 1 + f_i + K_i c}{m_i} \quad [c \in N_n(\mathbb{I})] \quad f_i \in \underbrace{\mathcal{I}_+ = \mathcal{I}}$
- Want to prove lct (Y, Γ) is c .
- Get rid of m_i by taking $\tilde{Y} \xrightarrow{\pi} Y$ ramified over Π_i of index m_i

$$K_{\tilde{Y}} + \tilde{\Gamma} = \pi^*(K_Y + \Gamma)$$

with $\text{coeff}(\tilde{\Gamma}_i) = f_i + K_i c$

- Now, $(\tilde{Y}, \sum_{\substack{i \\ \text{(H)}}} f_i \tilde{\Gamma}_i + c \sum_{\substack{i \\ M}} K_i \tilde{\Gamma}_i) = (\tilde{Y}, \Theta + cM)$ is lc but not klt
- c is the lct of (\tilde{Y}, Θ) w.r.t. the \mathbb{Z} -divisor M .
- $c \in LCT_{n+1}(\mathbb{I})$. 

Lemma 11.6: • (X, Δ) log CY, lc, $\dim n$

• $X = \mathbb{Q}$ -factorial

• $s(X) = 1$

$\text{coeff}(\Delta) \geq s \Rightarrow \Delta$ has at most $\frac{n+1}{s}$ components

How we use this: • We'll have sequence (X_i, Δ_i) with $\text{coeff}(\Delta_i) \geq s$

• Lemma \Rightarrow Can pass to subsequence s.t. #comp $(\Delta_i) = \text{constant}$.

11.7

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Prop 11.7: "Accumulation Points of $N_n(I) = N_{n-1}(I)$ "

- $I \subseteq [0, 1]$ s.t.
- $I = I_+$
- l is the only accum. pt. of I
- ($\Rightarrow l$ is the only accum. pt. of $D(I)$)

$$c_1 > c_2 > \dots \xrightarrow{\text{Limit}} c \neq 0 \quad c_i \in N_n(I)$$

(x_i, Δ_i) log CY, l_c , projective of dim n

$$\Delta_i = B_i + C_i$$

$$\sum_{\substack{i \\ \in D(I)}} \downarrow \quad \in D_{C_i}(I)$$

Then $c \in N_{n-1}(I)$.

(Stronger version for the purposes of induction:)

Assume instead: $\Delta_i = A_i + B_i + C_i$ B_i, C_i as before
 \downarrow
coeff approach |

Then $c \in N_{n-1}(I)$

Pf: Assume $[\Delta_i] = [A_i]$ [coeff of B_i, C_i bounded away from 1]

Let $a_i = \log \text{discrepancy } (x_i, \Delta_i)$

Case A: $\lim a_i > 0$ [Roughly speaking, coeff of Δ_i are bounded away from 1]
 A_i eventually zero $\log \text{disrep } (A_i) \rightarrow 0$

Step 1: Reduce to x_i Q-factorial, $s(x_i) = 1$

$(x_i, \Delta_i) \in \eta_n(I, c_i) \Rightarrow c_i \in N_n(I) = K_n(I)$

$\Rightarrow (x_i, \Delta_i) \in k_m(I, c_i)$ Q-factorial
 $s(x_i) = 1$

$\Rightarrow \exists j \dim x_i < n \forall i$, done by induction.

\Rightarrow Assume $m = n$

$\therefore 11.6 \Rightarrow$ Can pass to subseq. to assume #comp(B_i), C_i are fixed

coeff(B_i), coeff(C_i) bounded away from 1 + 1 is the only accum. point of $D(I)$,
 I_+

\therefore Can assume coeff(B_i) are fixed & coeff(C_i) looks like

$$\frac{r-1}{r} + \frac{1}{r} + \frac{k c_i}{r}$$

$$\frac{r-1}{r} + \frac{t}{r} + \frac{kc_i}{r}$$

Given $t \in [0,1]$, set $C_i(t) = \text{Same components as } C_i \text{ with coeff:}$

$$\frac{r-1}{r} + \frac{t}{r} + \frac{kt}{r}$$

$C_i(t)$ with $C_i(c_i) = C_i$

Let $h_i = \sup \{ t \mid (x_i, B_i + C_i(t)) \text{ is lc} \}$ be the lct

$$h_i \in LCT_n(D(I)) = N_{n-1}(D(I)) = N_{n-1}(I)$$

Set $h := \lim h_i$

Step 2: Reduce to $h > c$

In case $\underline{h \leq c_i}$, as $x_i, B_i + C_i(c_i)$ is lc, $c_i \leq h_i$

$$\begin{matrix} c_i \\ \downarrow \\ c \leq h \end{matrix}$$

$$\Rightarrow c = h \in N_{n-2}(I) \subseteq N_{n-1}(I)$$

Step 3: Reduce to $\text{vol}(x_i, \Delta_i)$ unbounded.

Suppose $\text{vol}(x_i, \Delta_i)$ bounded.

Can argue (x_i, Δ_i) log bir. bounded $\Rightarrow (x_i, \Delta_i)$ bounded family

$\therefore \exists$ ample divisor H_i s.t. $T_i \cdot H_i^{n-1}$ and $-K_{X_i} \cdot H_i^{n-1}$ bounded

(T_i is any comp. of Δ_i)

Pass to a subseq. and assume that the intersection numbers are const. Have:

$$(K_{x_i} + \Delta_i) \cdot H_i^{n-1} = 0 \quad \& \quad B_i \cdot H_i^{n-1} \text{ independent of } i$$

but $C_i \cdot H_i^{n-1}$ is not const., which is a contradiction.

Step 4: $\text{vol}(x_i, \Delta_i)$ unbounded

Use unboundedness of volume to modify x_i, Δ_i to produce S_i in Δ_i of coeff. 1. This therefore falls to Case B.

Case B: $\lim a_i = 0$ [\exists coeff of Δ_i which approach 1]

Step 1: Reduce to $A_i \neq 0$, X_i Q-factorial, (x_i, Δ_i) klt iff $[A_i] = 0$

dlt modification achieves this step. [Need to be careful about]

Step 2: Done if $[A_i] \cap C_i$ intersect.

Take comn C_i of $[A_i]$ which intersects C_i .

Step 2: Done if $[A_i] \& C_i$ intersect.

Take comp S_i of $[A_i]$ which intersects C_i .

$$(S_i, \mathbb{H}_i) \in \eta_{n-1}(I, C_i)$$

$$C_i \in N_{n-1}(I, C_i) \xrightarrow{\text{Induction}} C \in N_{n-2}(I) \subseteq N_{n-1}(I)$$

Step 3: Done if $f_i: X_i \rightarrow Z_i$ is a MFS,

A_i dominates Z_i ,

$$\dim Z_i > 0$$

$$(F_i, \mathbb{H}_i) = \text{General fiber}, \quad \mathbb{H}_i = A'_i + B'_i + C'_i$$

Case 1: $C'_i \neq 0$ Done by induction since $\dim F_i \leq n$

Case 2: $C'_i = 0$, C'_i does not dominate Z_i ,

$\therefore A_i, C_i$ intersect. Done by Step 2!

Step 4: Reduce to (X_i, Δ_i) klt

Suppose (X_i, Δ_i) not klt. Then $[A_i] \neq 0$.

Run $K_{X_i} + \Delta_i - [A_i]$ - MMP with scaling of some ample divisor to get (X'_i, Δ'_i)

If $C'_i = 0$, then $[A_i]$ and C_i intersect as step of MMP is $[A_i]$ -positive.

Done by Step 2!

If $C'_i \neq 0$, can proceed with MMP to end with MFS $X_i \xrightarrow{f} Z$

If $\dim Z > 0$, done by Step 3!

If $\dim Z = 0$, $g(x) = 1$, so $[A_i]$ and C_i intersect. Done by Step 2!

Step 5: Reduce to $g(x_i) = 1$

Run $K_{X_i} + \Delta_i - A_i$ - MMP and assume $\dim Z = 0$ (as $\dim Z > 0$ is Step 3)

Step 6: Finish Case B

As before, assume coeff (B_i) are fixed & coeff (C_i) look like:

$$\frac{r-1}{r} + \frac{t}{r} + \frac{k c_i}{r}$$

Set $C_i(t) = \text{Divisor with the same comp. as } C_i \text{ with coeff}$

$$\frac{r-1}{r} + \frac{k}{r} + \frac{kt}{r}$$

Let $T_i = \text{sum of components of } A_i$, but make all the coeffs 1.

$T_i \geq A_i$ [because $(x_i, A_i) = klt$ and so all coeff of $A_i < 1$.]

$C_i \geq C_i(c)$

$\therefore (x_i, A_i + B_i + C_i(c))$ is klt (as $(x_i, A_i + B_i + C_i)$ is klt)

Can pass to a subsequence to assume

$(x_i, T_i + B_i + C_i(c))$ is 1c

Suppose $(x_i, T_i + B_i + C_i)$ is not 1c.

Let $d_i = \sup \{t \in [c, c_i] \mid x_i, T_i + B_i + C_i(t) \text{ is } k\}$ be the lct

$d_i \in LCT_n(D(I)) = N_{n-1}(I)$ and $d_i \rightarrow c$ $\therefore c \in N_{n-2}(I)$ by induction.

\therefore Assume $(x_i, T_i + B_i + C_i)$ is 1c

Now, can show that pseff threshold $e_i = \sup \{t \mid K_{x_i} + T_i + B_i + C_i(t) \text{ pseff}\}$

can be shown to converge to c .

Either $e_i > e_{i+1} > \dots \rightarrow c$ or $e_i = c$ (after passing to a subseq.)

In this case, replace $C_i = C_i(c)$
by $C_i(e_i)$.

$\therefore \exists$ comp. of C_i intersecting T_i
and this reduces to Case B, Step 2.

In this case, restrict to a
comp. S_i of T_i and apply
adjunction to finish the proof.

